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TWO-LOOP SUPERSTRINGS III

Slice Independence and Absence of Ambiguities *

Eric D'Hoker[a] and D.H. Phong[b]

[a] *Department of Physics and*
Institute for Pure and Applied Mathematics (IPAM)
University of California, Los Angeles, CA 90095
[b] *Department of Mathematics*
Columbia University, New York, NY 10027

Abstract

The chiral superstring measure constructed in the earlier papers of this series for general gravitino slices ${\chi_{\bar z}}^+$ is examined in detail for slices supported at two points x_1 and x_2, ${\chi_{\bar z}}^+ = \zeta^1 \delta(z, x_1) + \zeta^2 \delta(z, x_2)$, where ζ^1 and ζ^2 are the odd Grassmann valued supermoduli. In this case, the invariance of the measure under infinitesimal changes of gravitino slices established previously is strengthened to its most powerful form: the measure is shown, point by point on moduli space, to be locally and globally independent from x_α, as well as from the superghost insertion points p_a, q_α introduced earlier as computational devices. In particular, the measure is completely unambiguous. The limit $x_\alpha = q_\alpha$ is then well defined. It is of special interest, since it elucidates some subtle issues in the construction of the picture-changing operator $Y(z)$ central to the BRST formalism. The formula for the chiral superstring measure in this limit is derived explicitly.

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1 Introduction

This paper is the third of a series whose goal is to show that two-loop amplitudes in superstring theory are fully slice-independent, do not suffer from any ambiguity, and can actually be expressed explicitly in terms of modular forms and sections of vector bundles over the moduli space of Riemann surfaces. The main formulas obtained have been announced in [1] (hereafter referred to as I), with the full derivations to appear in subsequent papers. In [2] (hereafter referred to as II), the first step has been carried out in detail, which is to derive from first principles a formula for the gauge-fixed amplitude, and to establish its invariance under infinitesimal changes of gauge slices.

The main purpose of the present paper III is to consider the family of worldsheet gravitino slices with support at two points x_α, $\alpha = 1, 2$,

$$\chi_{\bar z}{}^+ = \sum_{\alpha=1,2} \zeta^\alpha \delta(z, x_\alpha) \tag{1.1}$$

and to prove the full-fledged invariance, both under infinitesimal as well as global changes, of the gauge-fixed formula when restricted to this family. Gravitino slices of the form (1.1) are of great practical interest, since the gauge-fixed formula can be expressed then most simply in terms of meromorphic Green's functions and holomorphic differentials evaluated at x_α and the auxiliary superghost insertions q_α and p_a, $a = 1, 2, 3$. Our principal result in III is that the formula thus obtained is in fact *locally and globally independent of all the points x_α, q_α and p_a*. The independence from the points q_α and p_a was expected since these points were introduced merely as a computational device. The additional proof of their independence can be viewed as a check on the consistency of the entire approach and final formula. The independence from the points x_α is the crucial new fact, which really constitutes a proof that the chiral superstring measure is unambiguous and globally slice independent.

The independence from the points x_α, q_α, and p_a leads to a simpler gauge-fixed formula, where we can set $x_\alpha \to q_\alpha$. This simpler formula is also more natural, since the points p_a can be viewed as a slice for moduli, and the points q_α as a slice for supermoduli. More important, in the limit $x_\alpha \to q_\alpha$, the supercurrent insertions $S(x_\alpha)$ are made to coincide with the superghost insertions $\delta(\beta(q_\alpha))$. Formally, as shown in [3, 4], this product should yield the picture-changing operator $Y(z) = \delta(\beta(z))S(z)$ although the difficulties associated with taking $\delta(\beta(z))$ and $S(z)$ at coincident points had not been clarified before. A byproduct of our analysis is that indeed, the naive product of the supercurrent and the superghost insertions is then singular and ill-defined (c.f. Section 3.3 below). Including the subtle new contributions of our gauge-fixing procedure, however, the limit is automatically well-defined.

We obtain in this way what amounts to the correct(ed) prescription replacing that of the picture changing operator. The final formula is the main result of this paper,

$$\begin{aligned}
\mathcal{A}[\delta] &= i\mathcal{Z}\Big\{1+\mathcal{X}_1+\mathcal{X}_2+\mathcal{X}_3+\mathcal{X}_4+\mathcal{X}_5+\mathcal{X}_6\Big\} \\
\mathcal{Z} &= \frac{\langle \prod_a b(p_a)\prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det \omega_I\omega_J(p_a)}
\end{aligned} \tag{1.2}$$

and the $\mathcal{X}_i$ are given by

$$\begin{aligned}
\mathcal{X}_1+\mathcal{X}_6 &= \frac{\zeta^1\zeta^2}{16\pi^2}\Big[-10S_\delta(q_1,q_2)\partial_{q_1}\partial_{q_2}\ln E(q_1,q_2) \\
&\quad -\partial_{q_1}G_2(q_1,q_2)\partial\psi_1^*(q_2)+\partial_{q_2}G_2(q_2,q_1)\partial\psi_2^*(q_1) \\
&\quad +2G_2(q_1,q_2)\partial\psi_1^*(q_2)f^{(1)}_{3/2}(q_2)-2G_2(q_2,q_1)\partial\psi_2^*(q_1)f^{(2)}_{3/2}(q_1)\Big] \qquad (1.3)\\
\mathcal{X}_2 &= \frac{\zeta^1\zeta^2}{16\pi^2}\omega_I(q_1)\omega_J(q_2)S_\delta(q_1,q_2)\Big[\partial_I\partial_J\ln\frac{\vartheta[\delta](0)^5}{\vartheta[\delta](D_\beta)}+\partial_I\partial_J\ln\vartheta(D_b)\Big] \\
\mathcal{X}_3 &= \frac{\zeta^1\zeta^2}{8\pi^2}S_\delta(q_1,q_2)\sum_a\varpi_a(q_1,q_2)\Big[B_2(p_a)+B_{3/2}(p_a)\Big] \\
\mathcal{X}_4 &= \frac{\zeta^1\zeta^2}{8\pi^2}S_\delta(q_1,q_2)\sum_a\Big[\partial_{p_a}\partial_{q_1}\ln E(p_a,q_1)\varpi_a^*(q_2)+\partial_{p_a}\partial_{q_2}\ln E(p_a,q_2)\varpi_a^*(q_1)\Big] \\
\mathcal{X}_5 &= \frac{\zeta^1\zeta^2}{16\pi^2}\sum_a\Big[S_\delta(p_a,q_1)\partial_{p_a}S_\delta(p_a,q_2)-S_\delta(p_a,q_2)\partial_{p_a}S_\delta(p_a,q_1)\Big]\varpi_a(q_1,q_2)\,.
\end{aligned}$$

Here, the quantity $\partial\psi_1^*(q_2)$ is a tensor, given by

$$\partial\psi_1^*(q_2)=\frac{\vartheta[\delta](q_2-q_1+D_\beta)}{\vartheta[\delta](D_\beta)E(q_1,q_2)}\frac{\sigma(q_2)^2}{\sigma(q_1)^2}\,, \tag{1.4}$$

and we have

$$\begin{aligned}
f_n(w) &= \omega_I(w)\partial_I\ln\vartheta[\delta](D_n)+\partial_w\ln\Big(\sigma(w)^{2n-1}\prod_{i=1}^{2n-1}E(w,z_i)\Big) \qquad (1.5)\\
f^{(1)}_{3/2}(x) &= \omega_I(q_1)\partial_I\ln\vartheta[\delta](x+q_2-2\Delta)+\partial_{q_1}\ln\Big(E(q_1,q_2)E(q_1,x)\sigma(q_1)^2\Big) \\
f^{(2)}_{3/2}(x) &= \omega_I(q_2)\partial_I\ln\vartheta[\delta](x+q_1-2\Delta)+\partial_{q_2}\ln\Big(E(q_2,q_1)E(q_2,x)\sigma(q_2)^2\Big)\,.
\end{aligned}$$

The quantities B_2 and $B_{3/2}$ are defined as follows,

$$\begin{aligned}
B_2(w) &= -27T_1(w)+\frac{1}{2}f_2(w)^2-\frac{3}{2}\partial_w f_2(w)-2\sum_a\partial_{p_a}\partial_w\ln E(p_a,w)\varpi_a^*(w) \\
B_{3/2}(p_a) &= 12T_1(p_a)-\frac{1}{2}f_{3/2}(p_a)^2+\partial f_{3/2}(p_a)
\end{aligned} \tag{1.6}$$

All other quantities in (1.2) and (1.3) were defined in II, and will be discussed again below. Further background material can be found in [5, 6, 7].

2 Gravitino Slices Supported at Points

The starting point of this paper is the two-loop, even spin structure, chiral superstring measure derived in paper II for an arbitrary gravitino gauge slice

$$\chi_{\bar z}{}^{+} = \sum_{\alpha=1}^{2} \zeta^\alpha (\chi_\alpha)_{\bar z}{}^{+} \tag{2.1}$$

It is given by the following expression,

$$\begin{aligned}
\mathcal{A}[\delta] &= i\ \mathcal{Z}\Big\{1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6\Big\} \\
\mathcal{Z} &= \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det(\omega_I\omega_J(p_a)) \cdot \det\langle\chi_\alpha|\psi^*_\beta\rangle}
\end{aligned} \tag{2.2}$$

with the $\mathcal{X}_i$ defined as follows

$$\begin{aligned}
\mathcal{X}_1 &= -\frac{1}{8\pi^2}\int d^2z \chi_{\bar z}{}^{+} \int d^2w \chi_{\bar w}{}^{+} \langle S(z)S(w)\rangle \\
\mathcal{X}_2 &= +\frac{i}{4\pi}(\hat\Omega_{IJ} - \Omega_{IJ})\Big(5\partial_I\partial_J \ln\vartheta[\delta](0) - \partial_I\partial_J \ln\vartheta[\delta](D_\beta) + \partial_I\partial_J\ln\vartheta(D_b)\Big) \\
\mathcal{X}_3 &= +\frac{1}{2\pi}\int d^2z\hat\mu(w)\Big(B_2(w) + B_{3/2}(w)\Big) \\
\mathcal{X}_4 &= +\frac{1}{8\pi^2}\int d^2w\ \partial_{p_a}\partial_w \ln E(p_a,w)\chi_{\bar w}{}^{+}\int d^2u S_\delta(w,u)\chi_{\bar u}{}^{+}\varpi^*_a(u) \\
\mathcal{X}_5 &= +\frac{1}{16\pi^2}\int d^2u \int d^2v S_\delta(p_a,u)\chi_{\bar u}{}^{+}\partial_{p_a}S_\delta(p_a,v)\chi_{\bar v}{}^{+}\varpi_a(u,v) \\
\mathcal{X}_6 &= +\frac{1}{16\pi^2}\int d^2z \chi^*_\alpha(z)\int d^2w G_{3/2}(z,w)\chi_{\bar w}{}^{+}\int d^2v\chi_{\bar v}{}^{+}\Lambda_\alpha(w,v)
\end{aligned} \tag{2.3}$$

Explicit formulas are available in Appendix A of [2] for all the ingredients of this formula, such as the Green's functions for b,c ghosts $G_2(z,w)$ and for β,γ superghosts $G_{3/2}(z,w)$, for the prime form $E(z,w)$, for the Szegö kernel $S_\delta(z,w)$, and for the holomorphic 3/2 differentials ψ^*_α, normalized at points q_β by $\psi^*_\alpha(q_\beta) = \delta_{\alpha\beta}$. The Beltrani differential $\hat\mu$ effects the deformation of complex structures from the super period matrix to the supergeometry of the slice. We shall not repeat those definitions here, but refer the reader to [2] instead. There is an explicit formula available for the supercurrent correlator,

$$\begin{aligned}
\langle S(z)S(w)\rangle &= +\frac{5}{2}\partial_z\partial_w \ln E(z,w) S_\delta(z,w) \\
&\quad +\frac{3}{4}\partial_w G_2(z,w)G_{3/2}(w,z) + \frac{1}{2}G_2(z,w)\partial_w G_{3/2}(w,z) \\
&\quad -\frac{3}{4}\partial_z G_2(w,z)G_{3/2}(z,w) - \frac{1}{2}G_2(w,z)\partial_z G_{3/2}(z,w)\,.
\end{aligned} \tag{2.4}$$

Furthermore, B_2 and $B_{3/2}$ are holomorphic two forms, which are given by

$$B_2(w) = -27T_1(w) + \frac{1}{2}f_2(w)^2 - \frac{3}{2}\partial_w f_2(w) - 2\sum_a \partial_{p_a}\partial_w \ln E(p_a,w)\varpi_a^*(w)\,. \tag{2.5}$$

$$\begin{aligned} B_{3/2}(w) &= 12T_1(w) - \frac{1}{2}f_{3/2}(w)^2 + \partial_w f_{3/2}(w) \\ &\quad + \int d^2z\chi_\alpha^*(z)\Big(-\frac{3}{2}\partial_w G_{3/2}(z,w)\psi_\alpha^*(w) - \frac{1}{2}G_{3/2}(z,w)\partial_w\psi_\alpha^*(w) \\ &\qquad + G_2(w,z)\partial_z\psi_\alpha^*(z) + \frac{3}{2}\partial_z G_2(w,z)\psi_\alpha^*(z)\Big) \end{aligned} \tag{2.6}$$

Finally, χ_α^* are the linear combinations of χ_α normalized so that $\langle\chi_\alpha^*|\psi_\beta^*\rangle = \delta_{\alpha\beta}$ and the expression Λ_α is given by

$$\Lambda_\alpha(w,v) = 2G_2(w,v)\partial_v\psi_\alpha^*(v) + 3\partial_v G_2(w,v)\psi_\alpha^*(v) \tag{2.7}$$

2.1 The δ-function gravitino expressions for $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{X}_4$, $\mathcal{X}_5$

Considerable simplifications take place when the choice $\chi_\alpha(w) = \delta(w,x_\alpha)$ is made, even for points x_α that are arbitrary and unrelated to p_a or q_α. The limits of $\mathcal{X}_i$, $i=1,\cdots,5$ are manifestly regular, while that of $\mathcal{X}_6$ is regular only after careful manipulations.

We may readily evaluate the overall factor $\mathcal{Z}$ and the terms $\mathcal{X}_i$, $i=1,\cdots,5$ that enter into (2.2) and (2.3) on the gravitino slice $\chi_\alpha(w) = \delta(w,x_\alpha)$, and we find

$$\mathcal{Z} = \frac{\langle\prod_a b(p_a)\prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det\omega_I\omega_J(p_a)\cdot\det\psi_\beta^*(x_\alpha)} \tag{2.8}$$

$$\begin{aligned} \mathcal{X}_1 &= \frac{\zeta^1\zeta^2}{16\pi^2}\Big[-10S_\delta(x_1,x_2)\partial_{x_1}\partial_{x_2}\ln E(x_1,x_2) \\ &\qquad -3\partial_{x_2}G_2(x_1,x_2)G_{3/2}(x_2,x_1) - 2G_2(x_1,x_2)\partial_{x_2}G_{3/2}(x_2,x_1) \\ &\qquad +3\partial_{x_1}G_2(x_2,x_1)G_{3/2}(x_1,x_2) + 2G_2(x_2,x_1)\partial_{x_1}G_{3/2}(x_1,x_2)\Big] \\ \mathcal{X}_2 &= \frac{\zeta^1\zeta^2}{16\pi^2}\omega_I(x_1)\omega_J(x_2)S_\delta(x_1,x_2)\Big[\partial_I\partial_J\ln\frac{\vartheta[\delta](0)^5}{\vartheta[\delta](D_\beta)} + \partial_I\partial_J\ln\vartheta(D_b)\Big] \\ \mathcal{X}_3 &= \frac{\zeta^1\zeta^2}{8\pi^2}S_\delta(x_1,x_2)\sum_a\varpi_a(x_1,x_2)\Big[B_2(p_a)+B_{3/2}(p_a)\Big] \\ \mathcal{X}_4 &= \frac{\zeta^1\zeta^2}{8\pi^2}S_\delta(x_1,x_2)\sum_a\Big[\partial_{p_a}\partial_{x_1}\ln E(p_a,x_1)\varpi_a^*(x_2) + \partial_{p_a}\partial_{x_2}\ln E(p_a,x_2)\varpi_a^*(x_1)\Big] \\ \mathcal{X}_5 &= \frac{\zeta^1\zeta^2}{16\pi^2}\sum_a\Big[S_\delta(p_a,x_1)\partial_{p_a}S_\delta(p_a,x_2) - S_\delta(p_a,x_2)\partial_{p_a}S_\delta(p_a,x_1)\Big]\varpi_a(x_1,x_2)\,. \end{aligned} \tag{2.9}$$

Each expression is perfectly well-defined and each term is finite for generic points.

2.2 The δ-function gravitino expression for $\mathcal{X}_6$

In order to set $\chi_\alpha(w) = \delta(w, x_\alpha)$ in $\mathcal{X}_6$, we have to proceed with some extra care, since a singularity seems to emerge of the form $G_{3/2}(x_\alpha, x_\alpha)$ multiplying an expression trilinear in χ. This singularity is however only apparent, since it is naturally cancelled by the symmetry properties amongst the gauge slice functions χ_1 and χ_2 entering into this trilinear expression. The limit may then be taken safely and a good expression for $\mathcal{X}_6$ obtained. We present now a detailed account of this symmetrization and limiting process.

We begin by *keeping the slice functions χ_α arbitrary and regular*, letting them tend to δ-functions only after all singular contributions to $\mathcal{X}_6$ have cancelled out, and the limit can be taken safely. We need the following useful identity,

$$\sum_\alpha \chi^*_\alpha(z)\psi^*_\alpha(v) = \sum_\alpha \chi_\alpha(z)\bar\psi_\alpha(v)\,. \tag{2.10}$$

Here, $\psi^*_\alpha(v)$ are the holomorphic 3/2 differentials with normalization $\psi^*_\alpha(q_\beta) = \delta_{\alpha\beta}$, and χ^*_β are the linear combinations of χ_β dual to ψ^*_α, so that $\langle\chi^*_\alpha|\psi^*_\beta\rangle = \delta_{\alpha\beta}$. The holomorphic 3/2 differentials $\bar\psi_\alpha$ are then defined by (2.10), which implies $\bar\psi_\alpha(z)\langle\chi_\alpha|\psi^*_\beta\rangle = \psi^*_\beta(z)$, and this equation may be solved by

$$\begin{aligned}
\bar\psi_1(v) &= \frac{\psi^*_1(v)\langle\chi_2|\psi^*_2\rangle - \psi^*_2(v)\langle\chi_2|\psi^*_1\rangle}{\langle\chi_1|\psi^*_1\rangle\langle\chi_2|\psi^*_2\rangle - \langle\chi_2|\psi^*_1\rangle\langle\chi_1|\psi^*_2\rangle} \\
\bar\psi_2(v) &= \frac{\psi^*_2(v)\langle\chi_1|\psi^*_1\rangle - \psi^*_1(v)\langle\chi_1|\psi^*_2\rangle}{\langle\chi_1|\psi^*_1\rangle\langle\chi_2|\psi^*_2\rangle - \langle\chi_2|\psi^*_1\rangle\langle\chi_1|\psi^*_2\rangle}
\end{aligned} \tag{2.11}$$

As $\chi_\alpha(z) \to \delta(z, x_\alpha)$, $\bar\psi_\alpha$ has a smooth limit, the result becomes independent of the points q_α and normalized by $\bar\psi_\alpha(x_\beta) = \delta_{\alpha\beta}$. To evaluate the term $\mathcal{X}_6$, we first write all contributions in terms of χ_α instead of χ^*_α, using the above formula (2.10).

$$\begin{aligned}
\mathcal{X}_6 = \frac{\zeta^1\zeta^2}{16\pi^2}\int d^2z \int d^2w \int d^2v \Big[&+\chi_1(z)\chi_1(w)\chi_2(v)G_{3/2}(z,w)\bar\Lambda_1(w,v) \\
&-\chi_1(z)\chi_2(w)\chi_1(v)G_{3/2}(z,w)\bar\Lambda_1(w,v) \\
&+\chi_2(z)\chi_1(w)\chi_2(v)G_{3/2}(z,w)\bar\Lambda_2(w,v) \\
&-\chi_2(z)\chi_2(w)\chi_1(v)G_{3/2}(z,w)\bar\Lambda_2(w,v)\Big]
\end{aligned} \tag{2.12}$$

where $\bar\Lambda_\alpha$ is obtained by replacing ψ^*_α by $\bar\psi_\alpha$ in Λ_α of (2.7),

$$\bar\Lambda_\alpha(w,v) = 2G_2(w,v)\partial_v\bar\psi_\alpha(v) + 3\partial_v G_2(w,v)\bar\psi_\alpha(v) \tag{2.13}$$

The first term in (2.12) appears to generate a singularity $G_{3/2}(x_1, x_1)$ as $\chi_1(z) \to \delta(z, x_1)$. However, the simple pole of $G_{3/2}(z,w)$ is odd under the interchange of z and w, while the product $\chi_1(z)\chi_1(w)$ is even under this exchange. Thus, *for any regular* χ_1, *the pole term cancels* and the limit $\chi_1(z) \to \delta(z, x_1)$ is regular and may now be taken safely.

We begin by carrying out the symmetrization explicitly : in z and w in the first and fourth terms; in z and v in the second and third terms. Regrouping terms, we find

$$\begin{aligned}
\mathcal{X}_6 \; = \; & \frac{\zeta^1\zeta^2}{16\pi^2}\int d^2z \int d^2w \int d^2v \Big[\\
& +\frac{1}{2}\chi_1(z)\chi_1(w)\chi_2(v)\Big\{G_{3/2}(z,w)\bar\Lambda_1(w,v) + G_{3/2}(w,z)\bar\Lambda_1(z,v) \\
& \qquad -G_{3/2}(z,v)\bar\Lambda_1(v,w) - G_{3/2}(w,v)\bar\Lambda_1(v,z)\Big\} \\
& +\frac{1}{2}\chi_2(z)\chi_1(w)\chi_2(v)\Big\{G_{3/2}(z,w)\bar\Lambda_2(w,v) + G_{3/2}(v,w)\bar\Lambda_2(w,z) \\
& \qquad -G_{3/2}(z,v)\bar\Lambda_2(v,w) - G_{3/2}(v,z)\bar\Lambda_2(z,w)\Big\}\Big] .
\end{aligned} \tag{2.14}$$

Now we are ready to take the limit in which $\chi_\alpha(z) \to \delta(z, x_\alpha)$. In this limit, $w \to z$ in the first braces, while $v \to z$ in the second braces above. The terms that do not manifestly admit a limit may be evaluated with the help of the asymptotics of the Green function,

$$G_{3/2}(x,y) = \frac{1}{x-y} + f_{3/2}(x) + \mathcal{O}(x-y) \tag{2.15}$$

so that

$$\begin{aligned}
\lim_{w\to z}&\Big\{+G_{3/2}(z,w)\bar\Lambda_1(w,v) + G_{3/2}(w,z)\bar\Lambda_1(z,v)\Big\} \\
&\qquad = -\partial_z\bar\Lambda_1(z,v) + 2f_{3/2}(z)\bar\Lambda_1(z,v) \\
\lim_{v\to z}&\Big\{-G_{3/2}(z,v)\bar\Lambda_2(v,w) - G_{3/2}(v,z)\bar\Lambda_2(z,w)\Big\} \\
&\qquad = +\partial_z\bar\Lambda_2(z,w) - 2f_{3/2}(z)\bar\Lambda_2(z,w)
\end{aligned}$$

Next, we evaluate $\bar\Lambda_\alpha$ and the derivatives of $\bar\Lambda_\alpha$ needed in the above expressions as follows

$$\begin{aligned}
\bar\Lambda_1(x_1,x_2) &= 2G_2(x_1,x_2)\partial\bar\psi_1(x_2) \\
\bar\Lambda_2(x_2,x_1) &= 2G_2(x_2,x_1)\partial\bar\psi_2(x_1) \\
\partial_{x_1}\bar\Lambda_1(x_1,x_2) &= 2\partial_{x_1}G_2(x_1,x_2)\partial\bar\psi_1(x_2) \\
\partial_{x_2}\bar\Lambda_2(x_2,x_1) &= 2\partial_{x_2}G_2(x_2,x_1)\partial\bar\psi_2(x_1) \\
\bar\Lambda_1(x_2,x_1) &= 2G_2(x_2,x_1)\partial\bar\psi_1(x_1) + 3\partial_{x_1}G_2(x_2,x_1) \\
\bar\Lambda_2(x_1,x_2) &= 2G_2(x_1,x_2)\partial\bar\psi_2(x_2) + 3\partial_{x_2}G_2(x_1,x_2)
\end{aligned} \tag{2.16}$$

The quantities $\bar\psi_\alpha$ simplify in the limit $\chi_\alpha(z) \to \delta(z, x_\alpha)$ of (2.11) and the simplified expressions are given by

$$\begin{aligned}
\bar\psi_1(v) &= \frac{\psi_1^*(v)\psi_2^*(x_2) - \psi_2^*(v)\psi_1^*(x_2)}{\psi_1^*(x_1)\psi_2^*(x_2) - \psi_1^*(x_2)\psi_2^*(x_1)} \\
\bar\psi_2(v) &= \frac{\psi_2^*(v)\psi_1^*(x_1) - \psi_1^*(v)\psi_2^*(x_1)}{\psi_1^*(x_1)\psi_2^*(x_2) - \psi_1^*(x_2)\psi_2^*(x_1)}
\end{aligned} \tag{2.17}$$

We may now assemble all contributions into a final expression for $\mathcal{X}_6$,

$$\begin{aligned}
\mathcal{X}_6 = \frac{\zeta^1\zeta^2}{16\pi^2}\Big[& 3G_{3/2}(x_2,x_1)\partial_{x_2}G_2(x_1,x_2) - 3G_{3/2}(x_1,x_2)\partial_{x_1}G_2(x_2,x_1) \\
& +2G_{3/2}(x_2,x_1)G_2(x_1,x_2)\partial\bar\psi_2(x_2) - 2G_{3/2}(x_1,x_2)G_2(x_2,x_1)\partial\bar\psi_1(x_1) \\
& +2f_{3/2}(x_1)G_2(x_1,x_2)\partial\bar\psi_1(x_2) - 2f_{3/2}(x_2)G_2(x_2,x_1)\partial\bar\psi_2(x_1) \\
& +\partial_{x_2}G_2(x_2,x_1)\partial\bar\psi_2(x_1) - \partial_{x_1}G_2(x_1,x_2)\partial\bar\psi_1(x_2)\Big]
\end{aligned} \tag{2.18}$$

which is perfectly well-defined and finite.

It is worth pointing out that the sum $\mathcal{X}_1 + \mathcal{X}_6$ exhibits considerable simplification, as the terms multiplied by 3 occurring in $\mathcal{X}_6$ cancel those occurring in $\mathcal{X}_1$, and the total gives,

$$\begin{aligned}
\mathcal{X}_1 + \mathcal{X}_6 = \frac{\zeta^1\zeta^2}{16\pi^2}\Big[& -10S_\delta(x_1,x_2)\partial_{x_1}\partial_{x_2}\ln E(x_1,x_2) \\
& -2G_2(x_1,x_2)\partial_{x_2}G_{3/2}(x_2,x_1) + 2G_2(x_2,x_1)\partial_{x_1}G_{3/2}(x_1,x_2) \\
& +2G_{3/2}(x_2,x_1)G_2(x_1,x_2)\partial\bar\psi_2(x_2) - 2G_{3/2}(x_1,x_2)G_2(x_2,x_1)\partial\bar\psi_1(x_1) \\
& +2f_{3/2}(x_1)G_2(x_1,x_2)\partial\bar\psi_1(x_2) - 2f_{3/2}(x_2)G_2(x_2,x_1)\partial\bar\psi_2(x_1) \\
& +\partial_{x_2}G_2(x_2,x_1)\partial\bar\psi_2(x_1) - \partial_{x_1}G_2(x_1,x_2)\partial\bar\psi_1(x_2)\Big]
\end{aligned} \tag{2.19}$$

Together with the results of (2.9), the above formula yields the chiral superstring measure evaluated on δ-function supported gravitino slices.

3 Global Slice χ Independence

We shall now prove that the full chiral superstring measure $\mathcal{A}[\delta]$, given by (2.8), (2.9) and (2.18), is a holomorphic scalar function in x_α, q_α and p_a by showing that no singularities occur when any of these points pairwise coincide. Since the measure is a holomorphic scalar in x_1, for example, it must be independent of x_1. By iterating this argument for all points, we establish that $\mathcal{A}[\delta]$ is independent of all points x_α, q_α and p_a. Thus, $\mathcal{A}[\delta]$ is globally independent of the choice of δ-function slices. We present below the arguments for the absence of singularities when points coincide in order of increasing difficulty.

3.1 Regularity as $q_\alpha \to p_a$

This is the easiest case, as the overall factor $\mathcal{Z}$ as well as each term $\mathcal{X}_i$, $i = 1, \cdots, 6$ have a finite limit. This is manifest for all $\mathcal{X}_i$, except perhaps $\mathcal{X}_3$, where the result follows, however, from holomorphicity in w of the functions $B_2(w)$ and $B_{3/2}(w)$.

3.2 Regularity as $q_2 \to q_1$

The overall factor $\mathcal{Z}$

$$\mathcal{Z} = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det\omega_I\omega_J(p_a) \cdot \det\psi^*_\beta(x_\alpha)} \tag{3.1}$$

has q_α-dependence through both the correlator and the finite dimensional determinant $\det\psi^*_\beta(x_\alpha)$. The q-dependence of the latter may be exhibited using *any q_α-independent basis of 3/2 holomorphic differentials* ψ_1, ψ_2. We then have

$$\det\psi^*_\beta(x_\alpha) = \frac{\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)}{\psi_1(q_1)\psi_2(q_2) - \psi_1(q_2)\psi_2(q_1)} \tag{3.2}$$

The q_α-dependence of the correlator may also be rendered completely explicit,

$$\mathcal{Z} = \frac{\vartheta[\delta](0)^5 \vartheta(D_b) \prod_{a<b} E(p_a, p_b) \prod_a \sigma(p_a)^3}{Z^{15}\vartheta[\delta](q_1 + q_2 - 2\Delta)E(q_1, q_2)\sigma(q_1)^2\sigma(q_2)^2 \ \det\omega_I\omega_J(p_a) \cdot \det\psi^*_\beta(x_\alpha)} \tag{3.3}$$

The numerator of $\mathcal{Z}$ is q_α-independent. The denominator has a simple pole as $q_2 \to q_1$ from the factor $\det\psi^*_\beta(x_\alpha)$, and this pole is cancelled by a simple zero from the prime form $E(q_1, q_2)$, leaving a finite limit of $\mathcal{Z}$. The Green's functions $G_{3/2}(z, w)$ and $f_{3/2}(z)$ have a smooth limits as $q_2 \to q_1$, and these limits are given by (see Appendix A of [2])

$$\begin{aligned}
G_{3/2}(z, w) &= \frac{\vartheta[\delta](z - w + 2q_1 - 2\Delta)}{\vartheta[\delta](2q_1 - 2\Delta)E(z, w)} \frac{E(z, q_1)^2\sigma(z)^2}{E(w, q_1)^2\sigma(w)^2} \\
f_{3/2}(z) &= \omega_I(z)\partial_I \ln\vartheta[\delta](2q_1 - 2\Delta) + \partial_z \ln\Big(\sigma(z)^2 E(z, q_1)^2\Big)
\end{aligned} \tag{3.4}$$

As a result, $\mathcal{X}_1 + \mathcal{X}_6$ has a smooth limit, as do $\mathcal{X}_2$ and $\mathcal{X}_3$. The terms $\mathcal{X}_4$ and $\mathcal{X}_5$ are independent of q_α altogether, so their limit is manifestly smooth.

3.3 Regularity as $x_\alpha \to q_\alpha$

The limit $x_\alpha \to q_\alpha$ is not well-defined term by term, beginning with the superghost correlator $\mathcal{X}_1$. Here, we show that the combination of all contributions in (2.8), (2.9) and (2.18) is well-defined and finite. To begin with, it is manifest that the prefactor $\mathcal{Z}$ has a well-defined limit, with the only x_α dependence through $\det\psi^*_\beta(x_\alpha) \to 1$ in this limit. Furthermore, the terms $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{X}_4$ and $\mathcal{X}_5$ all have smooth limits. Thus, only the term $\mathcal{X}_1 + \mathcal{X}_6$ remains to be examined, which we do next.

The holomorphic differentials $\bar\psi_\alpha(z)$ behave smoothly as $x_\beta \to q_\beta$, as do their derivatives. Thus the first and last lines in (2.19) admit smooth limits, and only terms of the following form remain to be discussed,

$$G_2(x_1,x_2)\Big[-2\partial_{x_2}G_{3/2}(x_2,x_1) + 2G_{3/2}(x_2,x_1)\partial\bar\psi_2(x_2) + 2f_{3/2}(x_1)\partial\bar\psi_1(x_2)\Big] \qquad (3.5)$$

(minus the same form with x_1 and x_2 as well as $\bar\psi_1$ and $\bar\psi_2$ interchanged). The above contribution exhibits a singularity in the form of a simple pole in $x_1 - q_1$, but is regular as $x_2 \to q_2$. The pole is easily evaluated using te following formulas

$$\begin{aligned} G_{3/2}(x_2,x_1) &= \frac{1}{x_1 - q_1}\psi^*(x_2) + \mathcal{O}(1) \\ f_{3/2}(x_1) &= \frac{1}{x_1 - q_1} + \mathcal{O}(1) \end{aligned} \qquad (3.6)$$

The residue of the pole is given by

$$-2\partial\psi_1^*(x_2) + 2\psi_1^*(x_2)\partial\bar\psi_2(x_2) + 2\partial\bar\psi_1(x_2)\,, \qquad (3.7)$$

a formula in which $x_1 = q_1$ since we are evaluating the residue at the pole in $(x_1 - q_1)$. With this value for x_1, the $\bar\psi$ differentials (2.11) simplify considerably and we have

$$\bar\psi_1(x) = \psi_1^*(x) - \psi_2^*(x)\frac{\psi_1^*(x_2)}{\psi_2^*(x_2)} \qquad\qquad \bar\psi_2(x) = \frac{\psi_2^*(x)}{\psi_2^*(x_2)} \qquad (3.8)$$

Using these expressions, the residue is readily seen to vanish. We conclude that the limit $x_\alpha \to q_\alpha$ is smooth in the full chiral superstring measure.

3.4 Regularity as $p_a \to p_b$

The Green's function G_2 behaves smoothly in this limit, while S_δ, $G_{3/2}$ and $f_{3/2}$ are simply independent of p_a. As a result, $\mathcal{X}_1 + \mathcal{X}_6$ and $\mathcal{X}_2$ have smooth limits as two p_a collapse. The limits of $\mathcal{X}_3$, $\mathcal{X}_4$ and $\mathcal{X}_5$ are more involved as the forms ϖ_a^* and ϖ_a have implicit p_a

dependence which may become singular. To study this behavior, we fix $p_1 \neq p_2$ and let $p_3 \to p_1$ without loss of generality. The terms $\mathcal{X}_3$, $\mathcal{X}_4$ and $\mathcal{X}_5$ are now all of the form

$$\sum_a \varpi^*_a(x) f(p_a) \qquad\qquad \sum_a \varpi_a(x_1,x_2) f(p_a) \tag{3.9}$$

and we shall show that this limit is smooth provided f is differentiable, which is of course the case here. To analyze ϖ^*_a and ϖ_a in this configuration, it is convenient to choose an adapted basis for holomorphic Abelian differentials, $\omega^*_I(p_J) = \delta_{IJ}$ for $I, J = 1, 2$. We then have considerably simplified and more workable expressions for ϖ^*_a and ϖ_a, given by

$$\begin{aligned}
\varpi^*_1(x) &= \omega^*_1(x) - \frac{1}{2}\frac{\omega^*_1(p_3)}{\omega^*_2(p_3)}\omega^*_2(x) \\
\varpi^*_2(x) &= \omega^*_2(x) - \frac{1}{2}\frac{\omega^*_2(p_3)}{\omega^*_1(p_3)}\omega^*_1(x) \\
\varpi^*_3(x) &= \frac{1}{2}\Big(\frac{\omega^*_1(x)}{\omega^*_1(p_3)} + \frac{\omega^*_2(x)}{\omega^*_2(p_3)}\Big)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\varpi_1(x_1,x_2) &= \omega^*_1(x_1)\omega^*_1(x_2) - \frac{1}{2}\omega^*_1(p_3)\frac{\omega^*_1(x_1)\omega^*_2(x_2)+\omega^*_1(x_2)\omega^*_2(x_1)}{\omega^*_2(p_3)} \\
\varpi_2(x_1,x_2) &= \omega^*_2(x_1)\omega^*_2(x_2) - \frac{1}{2}\omega^*_2(p_3)\frac{\omega^*_1(x_1)\omega^*_2(x_2)+\omega^*_1(x_2)\omega^*_2(x_1)}{\omega^*_1(p_3)} \\
\varpi_3(x_1,x_2) &= \frac{1}{2}\frac{\omega^*_1(x_1)\omega^*_2(x_2)+\omega^*_1(x_2)\omega^*_2(x_1)}{\omega^*_1(p_3)\omega^*_2(p_3)}
\end{aligned} \tag{3.11}$$

The limits as $p_3 \to p_1$ of the sums are now easily evaluated and we find

$$\sum_a \varpi^*_a(x) f(p_a) = \frac{3}{2}\omega^*_1(x) f(p_1) + \omega^*_2(x) f(p_2) + \frac{1}{2}\frac{\omega^*_2(x)\partial f(p_1)}{\partial \omega^*_2(p_1)} \tag{3.12}$$

$$\begin{aligned}
\sum_a \varpi_a(x_1,x_2) f(p_a) &= \omega^*_1(x_1)\omega^*_1(x_2) f(p_1) + \omega^*_2(x_1)\omega^*_2(x_2) f(p_2) \\
&\quad + \frac{1}{2}\Big(\omega^*_1(x_1)\omega^*_2(x_2) + \omega^*_1(x_2)\omega^*_2(x_1)\Big)\frac{\partial f(p_1)}{\partial \omega^*_2(p_1)}
\end{aligned} \tag{3.13}$$

both of which are finite. This establishes that the limits of collapsing p_a's are smooth.

3.5 Regularity as $x_2 \to x_1$

The prefactor $\mathcal{Z}$ in (2.8) exhibits an overall simple pole as $x_2 \to x_1$ since the finite dimensional determinant $\det \psi^*_\beta(x_\alpha)$ has a simple zero in this limit. Amongst the $\mathcal{X}_i$ of (2.9), $\mathcal{X}_1$ exhibits a simple pole, which is cancelled by the simple poles in $\mathcal{X}_2$ and those parts of the simple pole in $\mathcal{X}_3$ that are produced by the full stress tensor. The remaining parts of $\mathcal{X}_3$ as well as $\mathcal{X}_4$ exhibit a simple pole, while $\mathcal{X}_5$ admits a vanishing limit.†

†It is helpful to notice that the prefactor $\mathcal{Z}$ as well as each $\mathcal{X}_i$ is odd under interchange of x_1 and x_2.

Only $\mathcal{X}_6$ appears to produce a triple pole, and we shall begin by showing that this pole cancels within $\mathcal{X}_6$. The starting point is the expression (2.18) and the limiting behaviors of $\partial\bar{\psi}_\alpha(x_\beta)$, given by

$$\begin{aligned} \partial\bar{\psi}_1(x_1),\ \partial\bar{\psi}_1(x_2) &\rightarrow \frac{1}{x_1 - x_2} \\ \partial\bar{\psi}_2(x_1),\ \partial\bar{\psi}_2(x_2) &\rightarrow \frac{-1}{x_1 - x_2} \end{aligned} \tag{3.14}$$

which upon substitution into (2.18) leads to the absence of the triple pole in $\mathcal{X}_6$.

3.5.1 Cancellation of the simple poles in $\mathcal{X}_i$

There are neither double poles nor constant terms in $\mathcal{X}_i$ since it is odd under $x_1 \leftrightarrow x_2$. Therefore, it remains to show that the simple poles cancel. To this end, an extra careful asymptotic analysis is required. We begin by defining the variables in which the limit will be taken :

$$x_1 = x + \epsilon \qquad\qquad x_2 = x - \epsilon \qquad\qquad \epsilon \rightarrow 0 \text{ with } x \text{ fixed}\,. \tag{3.15}$$

The derivatives $\partial\bar{\psi}_\alpha(x_\beta)$ behave as follows

$$\begin{aligned} \partial\bar{\psi}_1(x_1) &= +\frac{1}{2\epsilon} + \frac{1}{2}A + \frac{\epsilon}{2}B + \mathcal{O}(\epsilon^2) \\ \partial\bar{\psi}_1(x_2) &= +\frac{1}{2\epsilon} - \frac{1}{2}A + \frac{\epsilon}{4}(C - B) + \mathcal{O}(\epsilon^2) \\ \partial\bar{\psi}_2(x_1) &= -\frac{1}{2\epsilon} - \frac{1}{2}A + \frac{\epsilon}{4}(C - B) + \mathcal{O}(\epsilon^2) \\ \partial\bar{\psi}_2(x_2) &= -\frac{1}{2\epsilon} + \frac{1}{2}A - \frac{\epsilon}{2}B + \mathcal{O}(\epsilon^2) \end{aligned} \tag{3.16}$$

where A, B, C are defined by the following expressions, evaluated at x,

$$\begin{aligned} A &= \frac{\psi_2^*\partial^2\psi_1^* - \psi_1^*\partial^2\psi_2^*}{\psi_2^*\partial\psi_1^* - \psi_1^*\partial\psi_2^*} \\ B &= \frac{\frac{1}{3}\psi_2^*\partial^3\psi_1^* - \frac{1}{3}\psi_1^*\partial^3\psi_2^* + \partial\psi_2^*\partial^2\psi_1^* - \partial\psi_1^*\partial^2\psi_2^*}{\psi_2^*\partial\psi_1^* - \psi_1^*\partial\psi_2^*} \\ C &= \frac{\psi_2^*\partial^3\psi_1^* - \psi_1^*\partial^3\psi_2^* + \partial\psi_2^*\partial^2\psi_1^* - \partial\psi_1^*\partial^2\psi_2^*}{\psi_2^*\partial\psi_1^* - \psi_1^*\partial\psi_2^*}\,. \end{aligned} \tag{3.17}$$

We shall also need the asymptotics of the Green's functions, which are given as follows (see Appendix A of [2] for more details)

$$\partial_{x_1}\partial_{x_2}\ln E(x_1, x_2) = +\frac{1}{4\epsilon^2} - 2T_1(x) + \mathcal{O}(\epsilon^2)$$

$$
\begin{aligned}
S_\delta(x_1,x_2) &= +\frac{1}{2\epsilon} + 2\epsilon g_{1/2}(x) - 2\epsilon T_1(x) + \mathcal{O}(\epsilon^3) \\
G_n(x_1,x_2) &= +\frac{1}{2\epsilon} + f_n(x) + \epsilon(2g_n(x) - \partial f_n(x) - 2T_1(x)) + \mathcal{O}(\epsilon^2) \\
G_n(x_2,x_1) &= -\frac{1}{2\epsilon} + f_n(x) - \epsilon(2g_n(x) - \partial f_n(x) - 2T_1(x)) + \mathcal{O}(\epsilon^2) \\
\partial_{x_1} G_n(x_1,x_2) &= -\frac{1}{4\epsilon^2} + g_n(x) - T_1(x) + \mathcal{O}(\epsilon) \\
\partial_{x_2} G_n(x_2,x_1) &= -\frac{1}{4\epsilon^2} + g_n(x) - T_1(x) + \mathcal{O}(\epsilon)
\end{aligned} \tag{3.18}
$$

where $f_n(w)$ is given by the first line of (1.5) and $g_n(w)$ by

$$
g_n(w) = \frac{1}{2}\omega_I\omega_J(w)\partial_I\partial_J \ln\vartheta[\delta](D_n) + \frac{1}{2}f_n(w)^2 + \frac{1}{2}\partial_w f_n(w)\,. \tag{3.19}
$$

We now calculate the limiting pole behavior of each of the terms $\mathcal{X}_i$. We omit an overall factor of $\zeta^1\zeta^2/16\pi^2\epsilon$ which is common to all terms. The details of the calculation of $\mathcal{X}_3$ will be given below.

$$
\begin{aligned}
\mathcal{X}_1 + \mathcal{X}_6 &\sim 3g_2(x) + 12\,T_1(x) - 2\partial f_2(x) - Af_2(x) + \frac{1}{8}(3B+C) - 5g_{1/2}(x) \\
\mathcal{X}_2 &\sim \frac{1}{2}\omega\omega_J(x)\Big[\partial_I\partial_J \ln\frac{\vartheta[\delta](0)^5}{\vartheta[\delta](D_\beta)} + \partial_I\partial_J\ln\vartheta(D_b)\Big] \\
\mathcal{X}_3 &\sim -12\,T_1(x) + \frac{1}{2}f_2(x)^2 - \frac{3}{2}\partial f_2(x) - 2\sum_a \partial_{p_a}\partial_x \ln E(p_a,x)\varpi_a^*(x) - \frac{1}{2}f_{3/2}(x)^2 \\
&\quad -\frac{1}{2}\partial f_{3/2}(x) + g_{3/2}(x) + 4\partial f_2(x) - 4g_2(x) + Af_2(x) - \frac{1}{8}(3B+C) \\
\mathcal{X}_4 &\sim 2\sum_a \partial_{p_a}\partial_x \ln E(p_a,x)\varpi_a^*(x) \\
\mathcal{X}_5 &\sim 0
\end{aligned} \tag{3.20}
$$

Adding all terms but $\mathcal{X}_2$, and using the expressions for f_n of (1.5) and g_n of (3.19), we find

$$
\begin{aligned}
&\frac{1}{2}f_2(x)^2 - \frac{1}{2}f_{3/2}(x)^2 + \frac{1}{2}\partial f_2(x) - \frac{1}{2}\partial f_{3/2}(x) + g_{3/2}(x) - g_2(x) - 5g_{1/2}(x) \\
&= \frac{1}{2}\omega_I\omega_J(x)\Big[+\partial_I\partial_J\ln\vartheta[\delta](D_\beta) - \partial_I\partial_J\ln\vartheta(D_b) - 5\partial_I\partial_J\ln\vartheta[\delta](0)\Big]
\end{aligned} \tag{3.21}
$$

and this term is readily seen to cancel completely with $\mathcal{X}_2$.

3.5.2 Detailed evaluation of the limit of $\mathcal{X}_3$

The one piece of the above calculation that requires further detailing is the evaluation of $\mathcal{X}_3$. As $x_1 = x+\epsilon$ and $x_2 = x-\epsilon$, with x held fixed and $\epsilon \to 0$, the pole in $\mathcal{X}_3$ takes the following form

$$\text{pole } \mathcal{X}_3 = \frac{\zeta^1\zeta^2}{16\pi^2}\frac{1}{\epsilon}\sum_a \varpi_a(x,x)\Big(B_2(p_a)+B_{3/2}(p_a)\Big)\,. \tag{3.22}$$

First, we use the fact that $\varpi_a(x,x)=\phi_a^{(2)*}(x)$, and then we use the fact that since B_2 and $B_{3/2}$ are holomorphic 2-forms, we have

$$\sum_a \phi_a^{(2)*}(x)\Big(B_2(p_a)+B_{3/2}(p_a)\Big) = B_2(x)+B_{3/2}(x)\,. \tag{3.23}$$

The evaluation of $B_2(x)$ is straightforward. To evaluate $B_{3/2}(x)$, we keep $\epsilon\neq 0$, and work out its expression starting from its definition in (2.6). We find

$$\begin{aligned}
B_{3/2}(x) \;\; = \;\; & 12T_1(x)-\frac{1}{2}f_{3/2}(x)^2+\partial f_{3/2}(x)\\
& -\frac{3}{2}\partial_x G_{3/2}(x_1,x)\bar\psi_1(x)-\frac{3}{2}\partial_x G_{3/2}(x_2,x)\bar\psi_2(x)\\
& -\frac{1}{2}G_{3/2}(x_1,x)\partial\bar\psi_1(x)-\frac{1}{2}G_{3/2}(x_2,x)\partial\bar\psi_2(x)\\
& +G_2(x,x_1)\partial\bar\psi_1(x_1)+G_2(x,x_2)\partial\bar\psi_2(x_2)\\
& +\frac{3}{2}\partial_{x_1}G_2(x,x_1)+\frac{3}{2}\partial_{x_2}G_2(x,x_2)
\end{aligned} \tag{3.24}$$

To evaluate this quantity, we need further asymptotics of the Green's functions,[‡]

$$\begin{aligned}
G_{3/2}(x_1,x) &= +\frac{1}{\epsilon}+f_{3/2}(x)+\epsilon g_{3/2}(x)-\epsilon T_1(x)+\mathcal{O}(\epsilon^2)\\
G_{3/2}(x_2,x) &= -\frac{1}{\epsilon}+f_{3/2}(x)-\epsilon g_{3/2}(x)+\epsilon T_1(x)+\mathcal{O}(\epsilon^2)\\
\partial_x G_{3/2}(x_1,x) &= +\frac{1}{\epsilon^2}+\partial f_{3/2}(x)-g_{3/2}(x)+T_1(x)+\mathcal{O}(\epsilon)\\
\partial_x G_{3/2}(x_2,x) &= +\frac{1}{\epsilon^2}+\partial f_{3/2}(x)-g_{3/2}(x)+T_1(x)+\mathcal{O}(\epsilon)
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
G_2(x,x_1) &= -\frac{1}{\epsilon}+f_2(x)+\epsilon\partial f_2(x)-\epsilon g_2(x)+\epsilon T_1(x)+\mathcal{O}(\epsilon^2)\\
G_2(x,x_2) &= +\frac{1}{\epsilon}+f_2(x)-\epsilon\partial f_2(x)+\epsilon g_2(x)-\epsilon T_1(x)+\mathcal{O}(\epsilon^2)\\
\partial_{x_1}G_2(x,x_1) &= +\frac{1}{\epsilon^2}+\partial f_2(x)-g_2(x)+T_1(x)+\mathcal{O}(\epsilon)\\
\partial_{x_2}G_2(x,x_2) &= +\frac{1}{\epsilon^2}+\partial f_2(x)-g_2(x)+T_1(x)+\mathcal{O}(\epsilon)\,.
\end{aligned} \tag{3.26}$$

[‡]Notice that these limits are slightly different from those of (3.18) since here one of the arguments of the Green's functions is x in contrast with (3.18), where both arguments are x_α. As a result, the coefficients on the right are slightly different.

We also need new asymptotics of $\bar\psi_\alpha$ and its derivatives

$$\begin{aligned}
\bar\psi_1(x) &= +\frac{1}{2} - \frac{\epsilon}{4}A + \frac{\epsilon^2}{16}(-3B+C) + \mathcal{O}(\epsilon^3) \\
\bar\psi_2(x) &= +\frac{1}{2} + \frac{\epsilon}{4}A + \frac{\epsilon^2}{16}(-3B+C) + \mathcal{O}(\epsilon^3) \\
\partial\bar\psi_1(x) &= +\frac{1}{2\epsilon} - \frac{\epsilon}{16}(B+C) + \mathcal{O}(\epsilon^2) \\
\partial\bar\psi_2(x) &= -\frac{1}{2\epsilon} + \frac{\epsilon}{16}(B+C) + \mathcal{O}(\epsilon^2) \,.
\end{aligned} \tag{3.27}$$

Assembling all these pieces into the expression (3.24), we find the result given in (3.20).

3.6 Limits as $x_\alpha \to p_a$

We start from formulas (2.8) and (2.9) and evaluate the limit $x_\alpha \to p_a$ of each of the $\mathcal{X}_i$, $i = 1, \cdots, 6$. Since x_1 and x_2 play symmetrical roles, we examine only the limit $x_1 \to p_a$, without loss of generality. The only x_1-dependence of the prefactor (2.8) is through the finite-dimensional determinant $\det\psi^*_\beta(x_\alpha)$, which has a finite limit for generic points p_a. Thus, we shall need only the singular terms of the limit of $\mathcal{X}_i$ as $x_1 \to p_a$. We shall need the following asymptotics

$$\begin{aligned}
G_2(z,x_1) &= \frac{1}{x_1-p_a}\,\phi_a^{(2)*}(z) + \mathcal{O}(1) \\
\partial_{x_1}\partial_{p_a}\ln E(x_1,p_a) &= \frac{1}{(x_1-p_a)^2} + \mathcal{O}(1) \\
S_\delta(x_1,p_a) &= \frac{1}{x_1-p_a} + \mathcal{O}(x_1-p_a)
\end{aligned} \tag{3.28}$$

All other limits, up to regular terms, such as $\partial_{p_a}S_\delta(x_1,p_a)$ may be deduced from the above.

In evaluating the limits of the $\mathcal{X}_i$, there is an overall factor of $\zeta^1\zeta^2/16\pi^2$ which we shall suppress here. The various limits are then given by[§] on

$$\begin{aligned}
\mathcal{X}_1 &\sim -\frac{3}{(x_1-p_a)^2}\,\phi_a^{(2)*}(x_2)G_{3/2}(p_a,x_2) - \frac{1}{x_1-p_a}\,\phi_a^{(2)*}(x_2)\partial_{p_a}G_{3/2}(p_a,x_2) \\
\mathcal{X}_2 &\sim 0 \\
\mathcal{X}_3 &\sim -\frac{3}{(x_1-p_a)^2}\,S_\delta(p_a,x_2)\varpi_a^*(x_2) - \frac{3}{x_1-p_a}\partial_p\Big(S_\delta(p,x_2)\varpi_a(p,x_2)\Big)\Big|_{p=p_a} \\
&\quad +\frac{2}{x_1-p_a}\,S_\delta(p_a,x_2)\partial\bar\psi_1(x_1)\varpi_a^*(x_2)
\end{aligned}$$

[§]We use the notation $\partial_p X(p)\big|_{p=p_a}$ here and below whenever X has implicit dependence on p_a, such as is the case with $X = \varpi_a^*(p)$ and $X = \varpi_a(p,x_2)$.

$$
\begin{aligned}
\mathcal{X}_4 &\sim +\frac{2}{(x_1-p_a)^2}\; S_\delta(p_a,x_2)\varpi_a^*(x_2) + \frac{2}{x_1-p_a}\; \partial_{p_a} S_\delta(p_a,x_2)\varpi_a^*(x_2) \\
\mathcal{X}_5 &\sim +\frac{1}{(x_1-p_a)^2}\; S_\delta(p_a,x_2)\varpi_a^*(x_2) + \frac{1}{x_1-p_a}\; S_\delta(p_a,x_2)\; \partial_p \varpi_a(p,x_2)\Big|_{p=p_a} \\
&\quad -\frac{1}{x_1-p_a}\; \partial_{p_a} S_\delta(p_a,x_2)\varpi_a^*(x_2) \\
\mathcal{X}_6 &\sim +\frac{3}{(x_1-p_a)^2}\; \phi_a^{(2)*}(x_2) G_{3/2}(p_a,x_2) + \frac{3}{x_1-p_a}\; \phi_a^{(2)*}(x_2)\partial_{p_a} G_{3/2}(p_a,x_2) \\
&\quad -\frac{2}{x_1-p_a}\; \phi_a^{(2)*}(x_2) G_{3/2}(p_a,x_2)\partial\bar\psi_1(x_1) - \frac{2}{x_1-p_a} f_{3/2}(x_2)\phi_a^{(2)*}(x_2)\partial\bar\psi_2(x_1) \\
&\quad +\frac{1}{x_1-p_a}\partial\phi_a^{(2)*}(x_2)\partial\bar\psi_2(x_1)
\end{aligned}
\tag{3.29}
$$

It is easily established that the coefficients of all double poles simply cancel one another.

The remaining simple pole at $x_1 = p_a$ has the following residue which, after working out the p-derivative in $\mathcal{X}_3$ and regrouping terms, takes the form,

$$
\begin{aligned}
R_a(x_2;p,q) &= +2\phi_a^{(2)*}(x_2)\partial_{p_a} G_{3/2}(p_a,x_2) - 2\phi_a^{(2)*}(x_2) G_{3/2}(p_a,x_2)\partial\bar\psi_1(p_a) \\
&\quad -2\phi_a^{(2)*}(x_2) f_{3/2}(x_2)\partial\bar\psi_2(p_a) + \partial\phi_a^{(2)*}(x_2)\partial\bar\psi_2(p_a) \\
&\quad +2\varpi_a^*(x_2) S_\delta(p_a,x_2)\partial\bar\psi_1(p_a) - 2\varpi_a^*(x_2)\partial_{p_a} S_\delta(p_a,x_2) \\
&\quad -2S_\delta(p_a,x_2)\partial_p\varpi_a(p,x_2)|_{p=p_a}
\end{aligned}
\tag{3.30}
$$

It remain to show that this residue vanishes; the arguments are surprisingly involved.

The residue R_a is a differential form of weight $3/2$ in x_2. We begin by showing that this form has no poles in x_2, and is thus a holomorphic $3/2$ form. By inspecting each of the ingredients of R_a, it is clear that the only possible singularities in x_2 can occur at one of the points q_α or p_b. We show that there are in fact no such poles. Once this has been established, we shall show that the holomorphic $3/2$ form R_a vanishes at both points p_b, $b \neq a$. Since these points were generic, they cannot be the divisor of any non-vanishing holomorphic $3/2$ form, and thus $R_a = 0$.

3.6.1 Holomorphicity of the residue

To examine the singularity structure of R_a as $x_2 \to q_\alpha$, we need the following pole structure of each individual piece. First, $\phi^{(2)*}(x_2)$, $\partial\bar\psi_\beta(p_a)$, $\varpi_a^*(x_2)$ and $\partial_p\varpi_a(p,x_2)|_{p=p_a}$ are all regular in the limit. On the other hand, the singular ingredients are given by

$$
\begin{aligned}
G_{3/2}(p,x_2) &= \frac{1}{x_2-q_\alpha}\psi_\alpha^*(p) + \mathcal{O}(1) \\
f_{3/2}(x_2) &= \frac{1}{x_2-q_\alpha} + \partial\psi_\alpha^*(q_\alpha) + \mathcal{O}(x_2-q_\alpha)
\end{aligned}
\tag{3.31}
$$

Only the first three terms contribute to the pole, and the limiting behavior is given by

$$R_a = \frac{2\phi_a^{(2)*}(x_2)}{x_2 - q_\alpha}\Big(\partial\psi_\alpha^*(p_a) - \psi_\alpha^*(p_a)\partial\bar\psi_1(p_a) - \partial\bar\psi_2(p_a)\Big) + \mathcal{O}(1) \tag{3.32}$$

The definition of $\bar\psi_\beta$ in (2.17) involves ψ_γ^* as well as the values of x_1 and x_2. Since the residue R_a is evaluated at $x_2 = q_\alpha$, these definitions simplify considerably and may be conveniently expressed in terms of ψ_γ^*. Let $\alpha = 1$ without loss of generality,

$$\bar\psi_1(p) = \frac{\psi_2^*(p)}{\psi_2^*(x_1)} \qquad\qquad \bar\psi_2(p) = \psi_1^*(p) - \psi_2^*(p)\frac{\psi_1^*(x_1)}{\psi_2^*(x_1)} \tag{3.33}$$

for any point p. By differentiating in p and setting $p = x_1$, we get

$$\partial\bar\psi_1(x_1) = \frac{\partial\psi_2^*(x_1)}{\psi_2^*(x_1)} \qquad\qquad \partial\bar\psi_2(x_1) = \partial\psi_1^*(x_1) - \partial\psi_2^*(x_1)\frac{\psi_1^*(x_1)}{\psi_2^*(x_1)} \tag{3.34}$$

As a result, the factor in brackets in (3.32) vanishes for any point $x_1 = p_a$, $\partial\psi_\alpha^*(x_1) - \psi_\alpha^*(x_1)\partial\bar\psi_1(x_1) - \partial\bar\psi_2(x_1) = 0$ thereby showing that the pole of (3.32) is absent as $x_2 \to q_1$. The case $x_2 \to q_2$ is analogous. Thus, R_a has no poles as $x_2 \to q_\alpha$.

To examine the singularity structure as $x_2 \to p_b$, we have to deal with two distinct cases. When $b \neq a$, the first three terms in R_a tend to zero as $\phi_a^{(2)*}(p_b) = 0$, while the remaining terms have a finite limit. When $b = a$, double and single poles are generated,

$$R_a \sim \frac{4}{(x_2 - p_a)^2}\Big(-\phi_a^{(2)*}(x_2) + \varpi_a^*(x_2)\Big) + \frac{1}{x_2 - p_a}\Big(\partial_{x_2}\phi_a^{(2)*}(x_2) + 2\partial\varpi_a^*(p_a)\Big) \tag{3.35}$$

Expanding the argument x_2 around p_a in the double pole terms and using the identity

$$\partial_{x_2}\phi_a^{(2)*}(x_2)\Big|_{x_2=p_a} = 2\partial_{x_2}\varpi_a^*(x_2)\Big|_{x_2=p_a}$$

we see that this quantity cancels. Thus, the limit $x_2 \to p_b$ of R_a is regular as well.

3.6.2 Vanishing of the residue

It remains to show that $R_a = 0$ at the points $x_2 = p_b$ for $b \neq a$. The residue function simplifies at these values, and we have

$$\begin{aligned} R_a(p_b; p, q) \;\; = \;\; & +\partial\phi_a^{(2)*}(p_b)\partial\bar\psi_2(p_a) - 2S_\delta(p_a, p_b)\partial_p\varpi_a(p, p_b)\Big|_{p=p_a} \\ & +2\varpi_a^*(p_b)S_\delta(p_a, p_b)\partial\bar\psi_1(p_a) - 2\varpi_a^*(p_b)\partial_{p_a}S_\delta(p_a, p_b) \end{aligned} \tag{3.36}$$

As all points p_a are on an equal footing, we may choose, without loss of generality, $a = 3$ and $b = 1, 2$. It suffices to show that $R_3(p_1; p, q) = 0$; the same argument may then be applied to show that $R_3(p_2; p, q) = 0$ as well.

To demonstrate that $R_3(p_1;p,q)=0$, we evaluate $\phi_3^{(2)*}$, $\varpi_3^*(p_1)$ and $\partial\varpi_3(p_3,p_1)$ in a common basis where their expressions may be compared. To this end, we introduce (as in subsection §3.4) a basis of holomorphic Abelian differentials ω_I^* normalized so that $\omega_I^*(p_J)=\delta_{IJ}$, $I,J=1,2$. In terms of these objects, we have

$$\begin{aligned}
\partial\phi_3^{(2)*}(p_1) &= \frac{\partial\omega_2^*(p_1)}{\omega^*(p_3)\omega_2^*(p_3)} \qquad\qquad 2\partial\varpi_3^*(p_1)=\frac{\omega_2^*(p_3)}{\omega^*(p_3)\omega_2^*(p_3)}\\
2\partial_{p_3}\varpi_3(p_1,p_3) &= \frac{\partial\omega_2^*(p_3)}{\omega^*(p_3)\omega_2^*(p_3)}
\end{aligned}\tag{3.37}$$

Using these expressions, we may recast $R_3(p_1;p,q)$ in the following form,

$$\begin{aligned}
R_a(p_b;p,q) &= \frac{1}{\omega_2^*(p_3)}\Big(\partial\ln\omega_2^*(p_1)\partial\psi_1^*(p_3)+\partial\ln\omega_2^*(p_3)S_\delta(p_1,p_3)\\
&\qquad +\partial_{p_3}S_\delta(p_1,p_3)-\partial\psi_2^*(p_3)S_\delta(p_1,p_3)\Big)
\end{aligned}\tag{3.38}$$

The quantity in parentheses is in fact independent of p_2, as may be established by noticing that S_δ and $\partial\psi_1^*(p_2)$ are independnt of p_2, and that the remaining quantities are given by

$$\frac{\partial\omega_2^*(p_1)}{\omega_2^*(p_3)} = \frac{\vartheta(2p_1-w-\Delta)}{\vartheta(p_1+p_3-w-\Delta)}\frac{E(p_3,w)}{E(p_1,w)E(p_3,p_1)}\frac{\sigma(p_1)}{\sigma(p_3)}\tag{3.39}$$

$$\partial\ln\omega_2^*(p_3) = \omega_I(p_3)\partial_I\vartheta(p_1+p_3-w-\Delta)+\partial_{p_3}\ln\Big(\frac{E(p_3,p_1)\sigma(p_3)}{E(p_3,w)}\Big)\tag{3.40}$$

where w is an aribtrary point. Since $1/\omega_2^*(p_3)\neq 0$, showing the vanishing of $R_a(p_b;p,q)$ in (3.38) is equivalent to showing the vanishing of $\omega_2^*(p_3)R_a(p_b;p,q)$, which is just the bracket in (3.38). This quantity is a form of weight $1/2$ in p_1, and its only possible singularities are when $p_1\to p_3$. To show that this quantity vanishes, it suffices to show that it is holomorphic, since with even spin structure there are no holomorphic 1/2 forms. It suffices to pick up the poles as $p_1\to p_3$, which may be done with the help of

$$\begin{aligned}
\partial\psi_1^*(p_2) &\sim \frac{1}{p_1-p_2}-\omega_I(p_1)\partial_I\ln\vartheta[\delta](D_\beta)-2\partial_{p_1}\ln\sigma(p_1)\\
\frac{\partial\omega_2^*(p_1)}{\omega_2(p_2)} &\sim -\frac{1}{p_1-p_2}-\omega_I(p_1)\partial_I\ln\vartheta(2p_1-w_0-\Delta)-\partial_{p_1}\ln\sigma(p_1)+\partial_{p_1}\ln E(p_1,w_0)\\
\partial\ln\omega_2(p_2) &\sim -\frac{1}{p_1-p_2}+\omega_I(p_1)\partial_I\ln\vartheta(2p_1-w_0-\Delta)+\partial_{p_1}\ln\sigma(p_1)-\partial_{p_1}\ln E(p_1,w_0)\\
\partial\psi_2^*(p_2) &\sim \frac{1}{p_2-p_1}+\omega_I(p_1)\partial_I\ln\vartheta[\delta](D_\beta)+2\partial_{p_1}\ln\sigma(p_1)
\end{aligned}\tag{3.41}$$

and we see that all terms cancel. This concludes the proof of the fact that $R_a(x_2;p,q)=0$, and thus of the fact that the limits $x_\alpha\to p_a$ are regular.

4 The Limit $x_\alpha \to q_\alpha$ and Picture Changing Operators

The expression for the chiral superstring measure now involves 7 distinct generic points, x_α, q_α and p_a, upon which the actual amplitude does not depend. Clearly, one would like to do away with any reference to specific points in the final form of the chiral measure. One way to proceed is to let various points come together and collapse; all such limits are regular. Another way is to make special choices for the points without actually collapsing them. We shall make use of both approaches. We conclude this paper with the derivation of the final formulas (1.2) and (1.3), in which the points x_α and q_α have been collapsed onto one another and $x_\alpha = q_\alpha$. The resulting formula is the starting point of the next paper IV in this series, where the chiral superstring measure will be cast in terms of modular forms.

The limit $x_\alpha \to q_\alpha$ produces very significant simplifications such as $\det\psi^*_\beta(x_\alpha) = 1$, while $B_{3/2}(w)$ and some of the Green's functions simplify. The necessary ingredients are,

$$\begin{aligned}
G_{3/2}(x_2,x_1) &= \frac{1}{x_1-q_1}\psi_1^*(x_2) - \psi_1^*(x_2)f^{(1)}_{3/2}(x_2) + \mathcal{O}(x_1-q_1)\\
G_{3/2}(x_1,x_2) &= \frac{1}{x_2-q_2}\psi_2^*(x_1) - \psi_2^*(x_1)f^{(2)}_{3/2}(x_1) + \mathcal{O}(x_2-q_2)\\
f_{3/2}(x_1) &= \frac{1}{x_1-q_1} + \partial\psi_1^*(q_1) + \mathcal{O}(x_1-q_1)\\
f_{3/2}(x_2) &= \frac{1}{x_2-q_2} + \partial\psi_2^*(q_2) + \mathcal{O}(x_2-q_2)\,,
\end{aligned} \tag{4.1}$$

where we use the definitions of $f^{(\alpha)}_{3/2}(w)$, given in (1.5). Clearly, since the points p_a have been kept separate from the points q_α, the limits $x_\alpha \to q_\alpha$ on the Green function G_2 are regular. Similarly, the limit of the matter contribution is regular. The remaining contributions involve $G_{3/2}(x_1,x_2)$ and $G_{3/2}(x_2,x_1)$ respectively, of which the second is given by

$$-2\partial_{x_2}G_{3/2}(x_2,x_1) + 2G_{3/2}(x_2,x_1)\partial\bar\psi_2(x_2) + 2f_{3/2}(x_1)\partial\bar\psi_1(x_2)\,. \tag{4.2}$$

The $x_1 \to q_1$ limit of this quantity is regular, as was already shown in the preceding section, as a simple pole is cancelled between the three terms. The $x_2 \to q_2$ limit is regular for every term by itself, so both $x_\alpha \to q_\alpha$ limits are smooth and may be taken in any order. We begin by taking the limit $x_2 \to q_2$ first, which results in

$$-2\partial_{q_2}G_{3/2}(q_2,x_1) + 2f_{3/2}(x_1)\partial\bar\psi_1(q_2)\,, \tag{4.3}$$

since $G_{3/2}(q_2,x_1) = 0$. To take the limit $x_1 \to q_1$ next, we need to evaluate the factor $\partial\bar\psi_1(q_2)$, for which we use the fact that for $x_2 = q_2$, we have $\bar\psi_1(x) = \psi_1^*(x)/\psi_1^*(x_1)$, so that

$$\partial\bar\psi_1(q_2) = \partial\psi_1^*(q_2) - (x_1-q_1)\partial\psi_1^*(q_2)\partial\psi_1^*(q_1) + \mathcal{O}(x_1-q_1)^2\,. \tag{4.4}$$

Combining all, (4.3) is given by $2\partial\psi_1^*(q_2) f^{(1)}_{3/2}(q_2)$ and we find (1.2) with $\mathcal{X}_i$ given by (1.3).

In deriving the above limit, we started from a formula that had already combined the $\mathcal{X}_1$ and $\mathcal{X}_6$ contributions into their sum $\mathcal{X}_1+\mathcal{X}_6$, which admits a finite limit. It is, however, important to stress that neither $\mathcal{X}_1$ nor $\mathcal{X}_6$ by itself admits a limit as $x_\alpha \to q_\alpha$. Indeed, the limit of $\mathcal{X}_1$ would correspond to putting the supercurrent $S(x_\alpha)$ operator on top of the superghost insertion $\delta(\beta(q_\alpha))$; but this limit does not exist. Remarkably, the inclusion of the effects of the finite dimensional gauge fixing determinants which result in $\mathcal{X}_6$ render the limit well-defined. In particular, we obtain a well-defined interpretation of the picture changing operator $Y(z) = \delta(\beta(z))S(z)$. We view this intermediate result as one of the key successes of our approach.

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